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Exponential sums involving the Möbius function*

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1. INTRODUCTION

Let $\mu(n)$ denote the Möbius function and $e(x) = e^{2\pi ix}$ as usual. The exponential sum

$$S_k(x,\alpha) = \sum_{n \le x} \mu(n) e(n^k \alpha)$$

for $k = 1$ was first considered by Davenport [D1]. He proved by Vinogradov's elementary method that for any $A > 0$

$$\max_{\alpha \in [0,1]} |S_1(x,\alpha)| \ll_A x(\log x)^{-A}. \tag{1.1}$$

Here and in the sequel $\ll_A$ indicates that the implied constant depends at most on A. Unconditionally, there is no sharper estimate than (1.1) known today. However, if one assumes the generalized Riemann hypothesis, or briefly GRH, much better estimates can be obtained (see [HS] and [BH], for example). The best result in this direction is due to Baker and Harman [BH], who showed in 1991 that for any $\varepsilon > 0$

$$\max_{\alpha \in [0,1]} |S_1(x,\alpha)| \ll_\varepsilon x^{3/4+\varepsilon} \tag{1.2}$$

is valid under GRH.

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In the present paper we discuss the estimation of the exponential sum $S_k(x,\alpha)$. If we follow the proof of Davenport for (1.1), we can show that

$$\max_{\alpha\in[0,1]} |S_k(x,\alpha)| \ll_A x(\log x)^{-A}$$

holds for any $A>0$. Unfortunately however, a corresponding result of (1.2) for $S_k(x,\alpha)$ cannot be obtained in the same way as Baker and Harman in proving (1.2). In fact, GRH alone cannot even enable us to show that

$$\max_{\alpha\in[0,1]} |S_k(x,\alpha)| \ll x^{\theta}$$

for some constant $0<\theta<1$. This will be explained in the following section. The aim of this paper is to establish the following

Theorem. *For any $k\ge 2$ and $\varepsilon>0$, we have under GRH*

$$\max_{\alpha\in[0,1]} |S_k(x,\alpha)| \ll_\varepsilon x^{\varphi_k+\varepsilon},$$

where

$$\varphi_k = 1-\frac{1}{2^{2k-1}}.$$

2. OUTLINE OF THE PROOF

Take

$$P=x^{1/2}, \qquad Q=x^{k-1/2}.$$

It follows from Dirichlet's lemma on rational approximations that all $\alpha\in[0,1]$ can be divided into two disjoint sets

$$E_1=\left\{\alpha;\alpha=\frac{a}{q}+\lambda,(a,q)=1,1\le q\le P,|\lambda|\le\frac{1}{qQ}\right\},$$

$$E_2=\left\{\alpha;\alpha=\frac{a}{q}+\lambda,(a,q)=1,P<q\le Q,|\lambda|\le\frac{1}{qQ}\right\}.$$

For $\alpha\in E_1$ the exponential sum $S_k(x,\alpha)$ can be treated by analytic means under GRH. We shall prove in Section 3

Proposition 1. *Assume GRH. Then we have*

$$\max_{\alpha\in E_1}|S_1(x,\alpha)| \ll_\varepsilon x^{3/4+\varepsilon}, \tag{2.1}$$

and for $k\ge 2$,

$$\max_{\alpha\in E_1}|S_k(x,\alpha)| \ll_\varepsilon x^{1-1/(2k)+\varepsilon}. \tag{2.2}$$

What we shall actually prove is a more general result, namely, for $\alpha\in E_1$,

$$S_k(x,\alpha) \ll_\varepsilon q^{\eta_k}x^{1/2+\varepsilon}(1+|\lambda|^{1/2}x^{k/2}), \tag{2.3}$$

where

$$(2.4)\qquad \eta_k = \begin{cases} \frac{1}{2}, & \text{if } k = 1; \\ 1 - \frac{1}{k}, & \text{if } k \geq 2. \end{cases}$$

Proposition 1 follows easily from (2.3) and (2.4).

In the case $k = 1$, E_2 is empty, so that $[0,1] = E_1$. Thus the result (1.2) is a consequence of Proposition 1. While for $k \geq 2$, E_2 is nonempty, and GRH has no longer influence on the estimation of $S_k(x, \alpha)$ on E_2. In this case, one can establish

Proposition 2. *For $k \geq 2$ and $\alpha \in E_2$ we have unconditionally that*

$$\max_{\alpha \in E_2} |S_k(x,\alpha)| \ll_\varepsilon x^{1+\varepsilon}\left(\frac{1}{q} + \frac{1}{x^{1/2}} + \frac{q}{x^k}\right)^{4^{1-k}}.$$

This proposition follows from Vaughan's identity [Va] for $\mu(n)$ and the argument in [G] and [H], where it was proved, for $k = 2$ and general $k \geq 2$ respectively, that the same bound holds for the sum $S_k(x, \alpha)$ with $\mu(n)$ replaced by $\Lambda(n)$. We thus omit the proof of Proposition 2 and refer to [G] and [H] for details.

Clearly, the theorem may be derived directly from Proposition 1 and 2. In view of the gap between the results of the two propositions, we have good reason to expect that for $k \geq 2$,

$$\max_{\alpha \in [0,1]} |S_k(x,\alpha)| \ll_\varepsilon x^{1-1/(2k)+\varepsilon},$$

which might be shown by more efficient methods in treating $S_k(x, \alpha)$, $\alpha \in E_2$.

It should be mentioned here that the estimation of $S_k(x, \alpha)$ in short intervals has been investigated in [Z], [ZL1] and [ZL2], where it was proved unconditionally that

$$\max_{\alpha \in [0,1]} \left| \sum_{x < n \leq x+y} \mu(n) e(n^k \alpha) \right| \ll_{A,\varepsilon} yL^{-A}$$

is true for $x^{\theta_k + \varepsilon} \leq y \leq x$ with $\theta_1 = \frac{5}{8}$, $\theta_2 = \frac{11}{16}$, and $\theta_3 = \frac{4}{5}$, etc.

3. PROOF OF THE THEOREM

Lemma 1. *Suppose that $\chi_q = \chi_{\bmod q}$ is any character modulus q. Then for any integer $k \geq 1$ and $\varepsilon > 0$ we have that*

$$(3.1)\qquad \sum_{u=1}^{q} \chi(u) e\left(\frac{a}{q} u^k\right) \ll_{\varepsilon,k} (a,q)^{1/2} q^{1/2+\varepsilon}.$$

Proof. If $(a, q) = 1$, the estimation (3.1) is a well-known result of Vinogradov

[Vi]. The general case of (3.1) can also be proved in the same way as follows. Denote by I the left-hand side of (3.1). Then

$$I = \frac{1}{\varphi(q)} \sum_{u=1}^{q} \sum_{v=1}^{q} \chi(u)\chi(v) e\left(\frac{a}{q} u^k v^k\right)$$
$$= \frac{1}{\varphi(q)} \sum_{u=1}^{q} \sum_{v=1}^{q} \chi(x)\chi(y)\rho(x)\rho(y) e\left(\frac{a}{q} xy\right),$$

where $\rho(x)$ and $\rho(y)$ represent the number of solutions of the equation

$$u^k \equiv x \,(\mathrm{mod}\, q), \qquad v^k \equiv y \,(\mathrm{mod}\, q)$$

respectively. Since (see [Vi])

$$\rho(x), \rho(y) \ll q^{\varepsilon/2},$$

it follows that by Cauchy's inequality

$$|I|^2 \ll \frac{q^\varepsilon}{\varphi(q)} \sum_{x=1}^{q} \left| \sum_{y=1}^{q} \rho(x)\rho(y) e\left(\frac{a}{q} xy\right)\right|^2$$
$$= \frac{q^\varepsilon}{\varphi(q)} \sum_{y_1=1}^{q} \sum_{y_2=1}^{q} \rho(y_1)\chi(y_1)\rho(y_2)\bar{\chi}(y_2) \sum_{x=1}^{q} e\left(\frac{a(y_1-y_2)}{q} x\right)$$
$$\ll q^{2\varepsilon} \sum_{y_1=1}^{q} \sum_{\substack{y_2=1 \\ a(y_1-y_2)\equiv 0 \,(\mathrm{mod}\, q)}}^{q} 1$$
$$\ll q^{2\varepsilon} \sum_{y_1=1}^{q} \sum_{\substack{y_2=1 \\ y_1 \equiv y_2 \,(\mathrm{mod}\, q/(a,q))}}^{q} 1 \ll q^{2\varepsilon}(a,q)q.$$

Hence the lemma is proved. □

Lemma 2. *Let $\alpha = a/q + \lambda$, $(a,q) = 1$. Then for any $\varepsilon > 0$*

$$S_k(x,\alpha) \ll_\varepsilon q^{\eta_k+\varepsilon} \sum_{d \mid q} \max_{\chi_{qd^{-1}}} \left| \sum_{\substack{m \le xd^{-1} \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k \lambda)\right|,$$

where η_k is defined as in (2.4).

Proof. Working analogously to the proof of Lemma 1 of [BH], we have

$$S_k(x,\alpha) = \sum_{d \mid q} \frac{\mu(d)}{\varphi(qd^{-1})} \sum_{\chi_{qd^{-1}}} \left(\sum_{l=1}^{qd^{-1}} \bar{\chi}(l) e\left(\frac{ad^{k-1}}{qd^{-1}} l^k\right)\right)$$
$$\times \left(\sum_{\substack{m \le xd^{-1} \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k \lambda) \right),$$

which is, on appealing to Lemma 1,

$$\ll \sum_{d \mid q} \mu^2(d)(d^{k-1}, qd^{-1})^{1/2}(qd^{-1})^{1/2+\varepsilon} \max_{\chi_{qd^{-1}}}$$

$$\times \left| \sum_{\substack{m \le xd^{-1} \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k \lambda) \right|.$$

The result now follows from the elementary estimate that, for $d \mid q$

$$\mu^2(d)(d^{k-1}, qd^{-1})^{1/2}(qd^{-1})^{1/2} \ll q^{\eta_k}. \qquad \square$$

Lemma 3. *Suppose that $F(u)$ and $G(u)$ are real functions defined in $[a,b]$, $G(u)$ and $1/(F'(u))$ are monotonic.*

1. *If $|F'(u)| \gg m$ and $|G(u)| \ll M$, then*

$$\int_a^b G(u)e(F(u))\,du \ll \frac{M}{m}.$$

2. *If $|F''(u)| \gg r$ and $|G(u)| \ll M$, then*

$$\int_a^b G(u)e(F(u))\,du \ll \frac{M}{\sqrt{r}}.$$

See [T] Lemma 3.3 and Lemma 3.4.

Lemma 4. *Under GRH we have that*

$$\left| L\left(\frac{1}{2} + \varepsilon + it, \chi\right) \right|^{-1} \ll_{\varepsilon,\varepsilon'} (q(|t|+1))^{\varepsilon'}$$

for any $\varepsilon > 0$ and $\varepsilon' > 0$.

See [D2], for example.

We can now establish the theorem by giving

Proof of Proposition 1. By Lemma 2 we know that (2.3) will follow if we can prove that for any $\varepsilon > 0$ and $d \mid q$

$$\sum_{\substack{x/(2d) < m \le x/d \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k \lambda) \ll_\varepsilon d^{-1/2}x^{1/2+\varepsilon}(1 + |\lambda|^{1/2}x^{k/2}) \tag{3.2}$$

holds uniformly for all $\chi = \chi_{qd^{-1}}$.

Let I_1 denote the left-hand side of (3.2), and

$$F(s,\chi) = F_q(s,\chi) = \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \mu(m)\chi(m)m^{-s}, \quad \operatorname{Re} s > 1$$

$$H(s,\chi) = H_q(s,\chi) = \prod_{p \mid q} \left(1 - \frac{\chi(q)}{p^s}\right)^{-1}.$$

Then

(3.3) $$F(s,\chi) = L^{-1}(s,\chi)H(s,\chi).$$

By (3.3) we may conclude that under GRH the function $F(s,\chi)$ is analytic in the region $\operatorname{Re} s \geq \frac{1}{2} + \delta$ for any $\delta > 0$. Furthermore,

(3.4) $$H(s,\chi) \ll \prod_{p \mid q} \left(1 - \frac{1}{\sqrt{p}}\right)^{-1} \ll_\delta q^\delta, \quad \operatorname{Re} s \geq \frac{1}{2} + \delta.$$

By Perron's summation formula we have for $u \leq x$

$$\sum_{\substack{m \leq u \\ (m,q)=1}} \mu(m)\chi(m) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F(s,\chi) \frac{u^s}{s}\, ds + O\left(\frac{x^{1+\varepsilon}}{T} + \log x\right).$$

Take $T = x^k$ and shift the path of integration above to $\operatorname{Re} s = \frac{1}{2} + \frac{1}{2}\varepsilon$. It follows from Lemma 4 and (3.4) that

$$\sum_{\substack{m \leq u \\ (m,q)=1}} \mu(m)\chi(m) = \frac{1}{2\pi} \int_{-x^k}^{x^k} F\left(\frac{1}{2} + \frac{\varepsilon}{2} + it, \chi\right) \frac{u^{1/2+\varepsilon/2+it}}{\frac{1}{2} + \frac{\varepsilon}{2} + it}\, dt + O(x^\varepsilon).$$

Then

$$\begin{aligned}
I_1 &= \int_{x/(2d)}^{x/d} e(\lambda d^k u^k)\, d\left(\sum_{\substack{m \leq u \\ (m,q)=1}} \mu(m)\chi(m)\right) \\
&= \frac{1}{2\pi} \int_{-x^k}^{x^k} F\left(\frac{1}{2} + \frac{\varepsilon}{2} + it, \chi\right) dt \\
&\quad \times \int_{1/2xd^{-1}}^{xd^{-1}} u^{-1/2+\varepsilon/2} e\left(\lambda d^k u^k + \frac{t}{2\pi} \log u\right) du + O(|\lambda| x^{k+\varepsilon} + x^\varepsilon) \\
&\ll d^{-1/2} \int_{-x^k}^{x^k} \left| F\left(\frac{1}{2} + \frac{\varepsilon}{2} + it, \chi\right) \right| dt \\
&\quad \times \left| \int_{2^{-k}x^k}^{x^k} v^{-1+1/(2k)+\varepsilon/(2k)} e\left(\lambda v + \frac{t}{2k\pi} \log v\right) dv \right| + O(|\lambda| x^k + 1) x^\varepsilon.
\end{aligned}$$

Since

$$\left(\lambda v + \frac{t}{2k\pi} \log v\right)' = \frac{t + 2k\pi\lambda v}{2k\pi v} \gg \frac{\min_{2^{-k}x^k \leq v \leq x^k} |t + 2k\pi\lambda v|}{x^k}$$

$$-\left(\lambda v + \frac{t}{2k\pi} \log v\right)'' = \frac{t}{2k\pi v^2} \gg \frac{|t|}{x^{2k}},$$

by Lemma 3 we get

$$I_1 \ll d^{-1/2} x^{1/2+\varepsilon/2} \int_{-x^k}^{x^k} \left| F\left(\frac{1}{2}+\frac{\varepsilon}{2}+it, \chi\right) \right|$$

$$\times \min\left(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{2^{-k}x^k \le v \le x^k} |t+2k\pi\lambda v|}\right) dt + (|\lambda| x^k + 1) x^{\varepsilon}$$

$$\ll d^{-1/2} x^{1/2+(3\varepsilon)/4} \int_{-x^k}^{x^k}$$

$$\times \min\left(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{2^{-k}x^k \le v \le x^k} |t+2k\pi\lambda v|}\right) dt + (|\lambda| x^k + 1) x^{\varepsilon}.$$

In the last step we have applied (3.4) and Lemma 4 once more. On noting that

$$|\lambda| x^k \le d^{-1/2} |\lambda|^{1/2} x^{(k+1)/2}$$

it suffices now to show that

$$(3.5) \quad \begin{cases} \displaystyle\int_{-x^k}^{x^k} \min\left(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{2^{-k}x^k \le v \le x^k} |t+2k\pi\lambda v|}\right) dt \\ \qquad \ll (1+|\lambda|^{1/2} x^{k/2}) \log x. \end{cases}$$

Denote by I_2 the left-hand side of (3.5). If $|\lambda| > x^{-k}$, then

$$I_2 \ll \int_{|t| \le 2^{-k}\pi|\lambda| x^k} \frac{dt}{|\lambda| x^k} + \int_{4k\pi|\lambda| x^k < |t| \le x^k} \frac{dt}{|t|}$$

$$+ \int_{2^{-k}\pi|\lambda| x^k < |t| \le 4k\pi|\lambda| x^k} \frac{dt}{\sqrt{|t|+1}}$$

$$\ll \log x + |\lambda|^{1/2} x^{k/2}.$$

If $|\lambda| \le x^{-k}$, we have that

$$I_2 \ll \int_{|t| \le 4k\pi} 1\, dt + \int_{4k\pi < |t| \le x^k} \frac{dt}{|t|} \ll \log x.$$

This proves (3.5), and the result follows. □

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